[146141] / SS 2025
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Calculus 2

Exercise Sheet 10

To be discussed on **19.05.2025**

Exercise 10.1 - The Divergence Theorem.
Consider the vector field $F(x, y, z) = (0, 0, z^2)$ and the surface Σ given by the whole surface of the cone $x^2 + y^2 = (1 - z)^2$, $0 \leq z \leq 1$, including the base $x^2 + y^2 = 1$, $z = 0$ (ES 9, exercise 9.6). Exploit the Divergence Theorem to show that

$$\int_\Sigma F \cdot N \mathrm{d}s = \frac{\pi}{6}.$$

Exercise 10.2 - An important consequence of the Divergence Theorem.
A closed region Ω is bounded by a simple surface Σ. Use the Divergence Theorem to show that

$$\int_\Omega \nabla\phi \cdot \nabla\psi \, \mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_\Sigma \phi \frac{\partial \psi}{\partial n} \, \mathrm{d}S - \int_\Omega \phi \Delta\psi \, \mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

where ϕ and ψ are scalar functions, n is the outer normal vector, and Δ the Laplace operator.
Hint: Compute $\operatorname{div}(\phi\nabla\psi)$ *first. Remember that in 3D*

$$\Delta\psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}.$$

Exercise 10.3 - Stokes theorem and the circulation of a vector field.
Compute the circulation of the vector field $F(x, y, z) := (z - y, x - z, y - x)$, along the boundary of the triangle of vertexes $A = (1, 0, 0)$, $B = (0, 1, 1)$ and $C = (0, 0, 1)$ oriented in such a way that one moves from A towards B, then towards C and back to A. Verify the Stokes theorem.